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OPTIMUM AND SUB-OPTIMUM TECHNIQUES FOR EXTRACTION OF INPUT SIGNALS FROM THE OUTPUTS OF NOISY TIME-VARIABLE MULTIPATH CHANNELS

by

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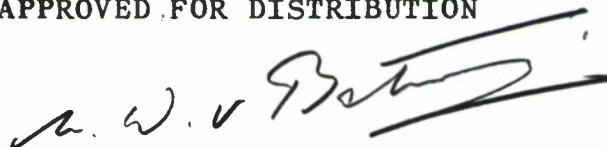I 19026 - La Spezia, Italy

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OPTIMUM AND SUB-OPTIMUM TECHNIQUES FOR EXTRACTION
OF INPUT SIGNALS FROM THE OUTPUTS OF NOISY
TIME-VARIABLE MULTIPATH CHANNELS

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F. Bryn

ABSTRACT

In underwater communication and echo-ranging the transmitted
signal will generally reach the receiver via several distinct
paths. The resultant signal is referred to as a multipath signal.
The first part of this report derives the integral equation that
defines the optimum filter for extracting the transmitted signal
from a noise-corrupted multipath signal. The solution of this
equation is not considered to be easily accessible. The second
part of the report describes a practical technique for deriving
near-optimum extraction filters applicable to echo-ranging
situations involving reflections from a smooth ocean surface.

INTRODUCTION

Consider the randomly time-varying multipath transmission channel depicted in Fig. 1. A transmitted signal s(t) gives rise to a
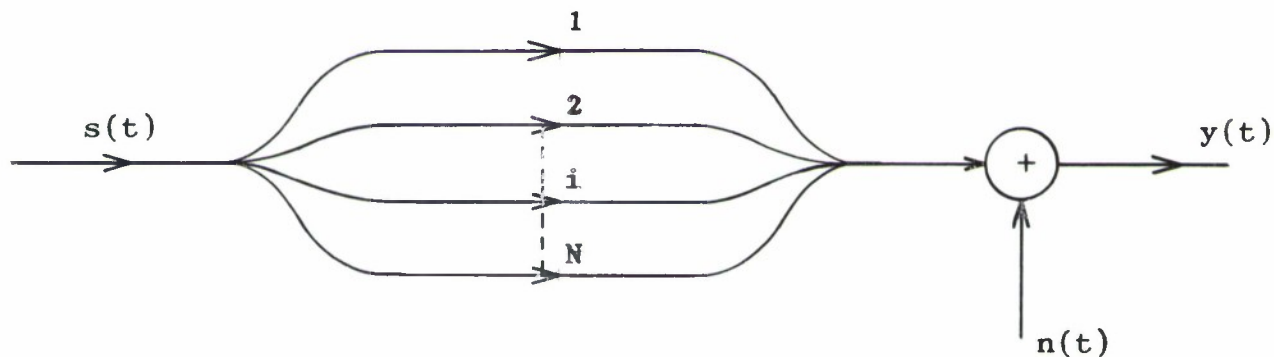
FIG. 1 NOISE-CORRUPTED MULTIPATH CHANNEL

noise-corrupted multipath signal

$$y(t) = \sum_{i=1}^{N} \int_{0}^{\infty} h_i(\tau,t) \, s(t-\tau) \, d\tau + n(t) \qquad (Eq. 1)$$

in which $h_i(\tau,t)$ is the response of the i'th channel, at time t, to a unit impulse applied τ seconds earlier.

The present report investigates the problem of defining the optimum linear filter for extracting the transmitted signal s(t) from the

received waveform y(t). It was considered to be of particular interest to study extraction techniques applicable to multipath explosive echo-ranging situations in which it is desirable to isolate the response of the target to the explosive pulse.

The similarity between this problem and the pure transmission problem presented above is best understood by considering Fig. 2a.
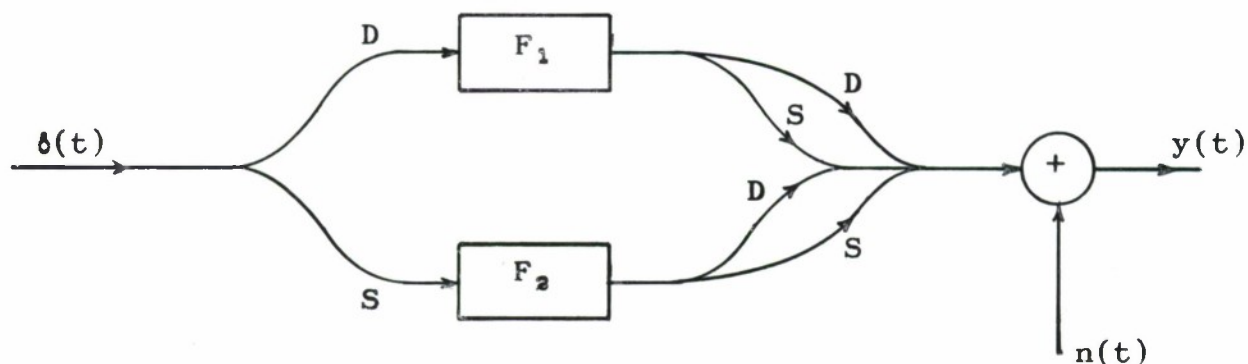

FIG. 2a EQUIVALENT DIAGRAM FOR MULTIPATH ECHO-RANGING

An explosive type pulse, in the figure denoted by $\delta(t)$, is transmitted from a point in the ocean at time $t = 0$. The pulse travels via one direct path (D) and one surface-reflected path (S) to the target represented by the two filters F_1 and F_2. The output signals from the filters, being the responses of the target to the two incoming pulses, are reflected towards the receiver, each via one direct and one surface-reflected path. If the impulse responses of the filters are denoted by $s_1(t)$ and $s_2(t)$ respectively, Fig. 2a can be redrawn as shown in Fig. 2b. Finally,

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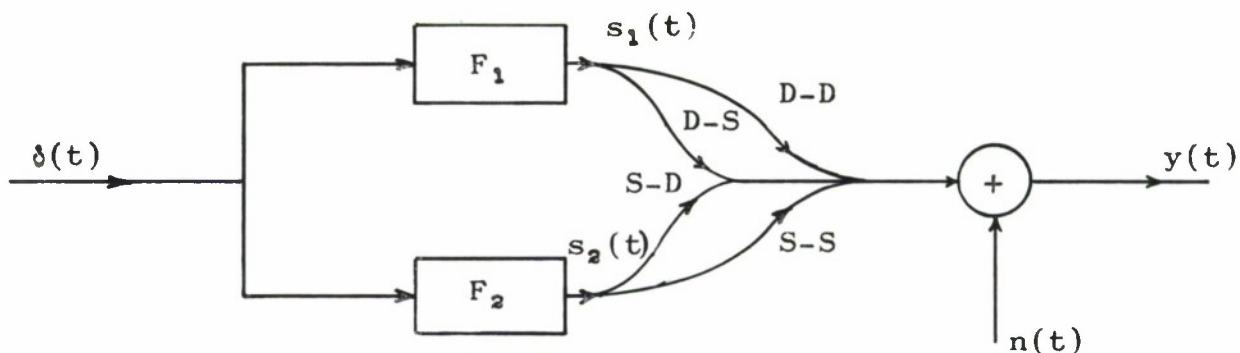
FIG. 2b   EQUIVALENT DIAGRAM FOR MULTIPATH ECHO-RANGING

if $s_1(t) = s_2(t) = s(t)$ the diagram becomes as shown in Fig. 2c.
This is similar to Fig. 1 with $N = 4$.
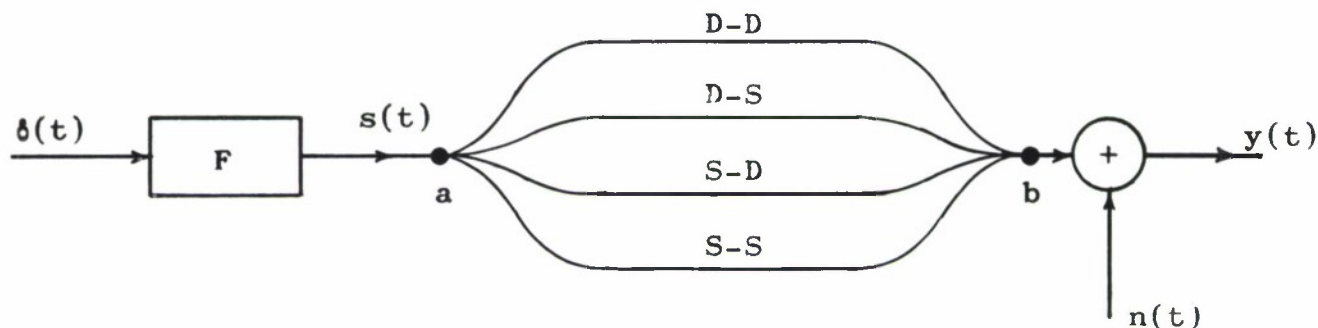
FIG. 2c   FINAL EQUIVALENT DIAGRAM

The results obtained are twofold. Firstly, the integral equation
governing the form of the optimum linear extraction filter for the
noise-corrupted, randomly time-varying, multipath channel is
derived; the solution of this equation is not believed to be easily
accessible. Secondly, a practical, intuitive technique for

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deriving near-optimum extraction filters for multipath, noise-corrupted, echo-ranging situations involving reflections from a smooth surface, is considered. The time-varying element is here eliminated and only the noise retained.

1. THE IMPULSE RESPONSE OF THE CHANNEL

From Eq. 1 the time-varying impulse response of the channel is

$$h(\tau, t) = \sum_{i=1}^{N} h_i(\tau, t) \; , \qquad\qquad\qquad \text{(Eq. 2)}$$

where $h_i(\tau, t)$ defines the response of the i'th channel, at time t, to a unit impulse applied τ seconds earlier. In physical systems we always have

$$h_i(\tau, t) = 0 \quad \text{for} \quad \tau < 0$$

The Fourier transform of $h(\tau, t)$ with respect to τ is the time-varying frequency response $H(f, t)$ of the channel. Thus

$$H(f, t) = \int_{0}^{\infty} h(\tau, t) \; e^{-j2\pi f\tau} \; d\tau$$

and

$$\text{(Eq. 3)}$$

$$h(\tau, t) = \int_{-\infty}^{+\infty} H(f, t) \; e^{j2\pi f\tau} \; df \; .$$

In what follows we shall assume that $h(\tau, t)$ and $H(f, t)$ are random functions of time t and that they are members of stationary processes $h(\tau, t)$ and $H(f, t)$ respectively.

2. THE OPTIMUM EXTRACTION FILTER

Referring to Eq. 1, the problem considered is as follows:
Given a mixture

$$y(t) \; = \; m(t) + n(t) \qquad \qquad (Eq. \; 4)$$

of a multipath signal

$$m(t) \; = \; \sum_{i=1}^{N} \int_{0}^{\infty} h_i(\tau, t) \cdot s(t - \tau) \; d\tau \qquad (Eq. \; 5)$$

and stationary noise $n(t)$, what is the optimum, physically-realizable,
linear filter for estimating $s(t - t_o)$ from $y(t)$? The delay t_o
may be arbitrarily selected by the filter designer.

It is assumed that the transmitted signal $s(t)$ has finite
length T and that we only are interested in the filter estimator
output in the interval from t_o to $(t_o + T)$.

Let $g(\tau)$ be the impulse response of the filter estimator and
$Z(t)$ its output, as shown in Fig. 3. Mathematically our problem

$$y(t)=m(t)+n(t) \longrightarrow \boxed{g(\tau)} \dashrightarrow \begin{array}{l} \text{Actual output} = z(t) \\ \text{Desired output} = s(t - t_o) \end{array}$$

FIG. 3

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is identical to that of determining $g(\tau)$ such that the mean square error:

$$\mu_g \;=\; E \int_{t_o}^{t_o+T} \Big[z(t) - s(t-t_o) \Big]^2 \, dt \qquad\qquad (\text{Eq. 6})$$

is a minimum. Note that

$$z(t) \;=\; \int_0^\infty g(\tau) \cdot y(t-\tau) \, d\tau \; .$$

The minimization may be done by variational techniques, i.e. one introduces a new filter estimator with impulse response

$$k(\tau) \;=\; g(\tau) + \epsilon \cdot \eta(\tau)$$

and requires that the derivative of the corresponding mean square error with respect to ϵ be zero for $\epsilon = 0$ for all functions $\eta(\tau)$. It is assumed that the properties of $\eta(\tau)$ are those of a physically realizable filter response.

The minimization process is shown formally in Appendix A. The result is the integral equation:

$$\int_0^\infty g(\tau) \Big[\varphi_{m.m}(\tau,v) + T \cdot \varphi_{n.n}(\tau-v) \Big] \cdot d\tau - \varphi_{m.s}(t_o,v) = 0 \qquad \underline{v > 0}$$

$$(\text{Eq. 7})$$

where

$$\varphi_{m.m}(\tau,v) = \int_{t_o}^{t_o+T} E\left(m(t-\tau) \cdot m(t-v)\right) \; dt,$$

$$\varphi_{n.n}(\tau-v) = E\left(n(t-\tau) \cdot n(t-v)\right) \; ,$$

$$\varphi_{m.s}(t_o,v) = \int_{t_o}^{t_o+T} s(t-t_o) \cdot E\left(m(t-v)\right) dt \; ,$$

The function $m(t)$ is given by Eq. 5. Figures 4a, b, c, and d present some simplified illustrations of the signals involved in Eq. 7.

If t_o is chosen sufficiently large, all terms on the lefthand side of Eq. 7 will approach zero for $v < 0$. It is then possible to apply a Fourier transform with respect to the variable v. Introducing

$$G(f) = \int_0^\infty g(\tau) \; e^{-j2\pi f \tau} \; d\tau \; ,$$

$$N(\xi) = \int_{-\infty}^{+\infty} \varphi_{nn}(v) \; e^{-j2\pi \cdot \xi \cdot v} \; dv \; ,$$

$$\Phi_{m.m}(f,\xi) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi_{m.m}(\tau,v) \; e^{j2\pi(f\cdot\tau - \xi\cdot v)} \; d\tau \cdot dv \; ,$$

and

$$\Phi_{m.s}(t_o,\xi) = \int_{-\infty}^{+\infty} \varphi_{m.s}(t_o,v) \; e^{-j2\pi\cdot\xi\cdot v} \; dv \; ,$$

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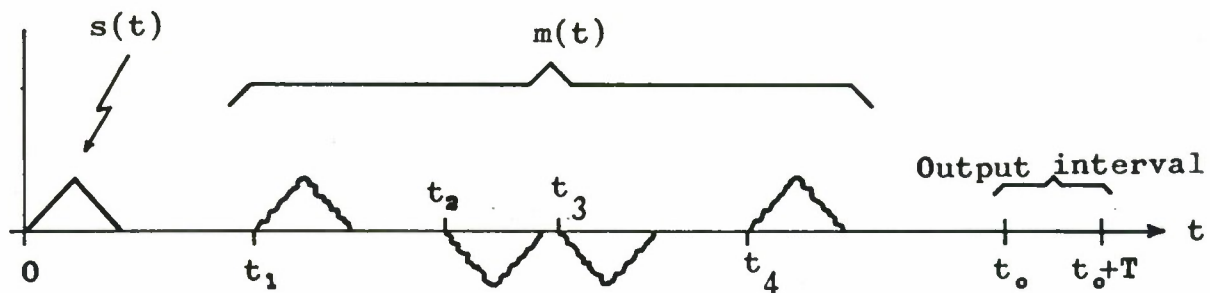FIG. 4a   TRANSMITTED SIGNAL s(t) AND MULTIPATH OUTPUT   m(t)
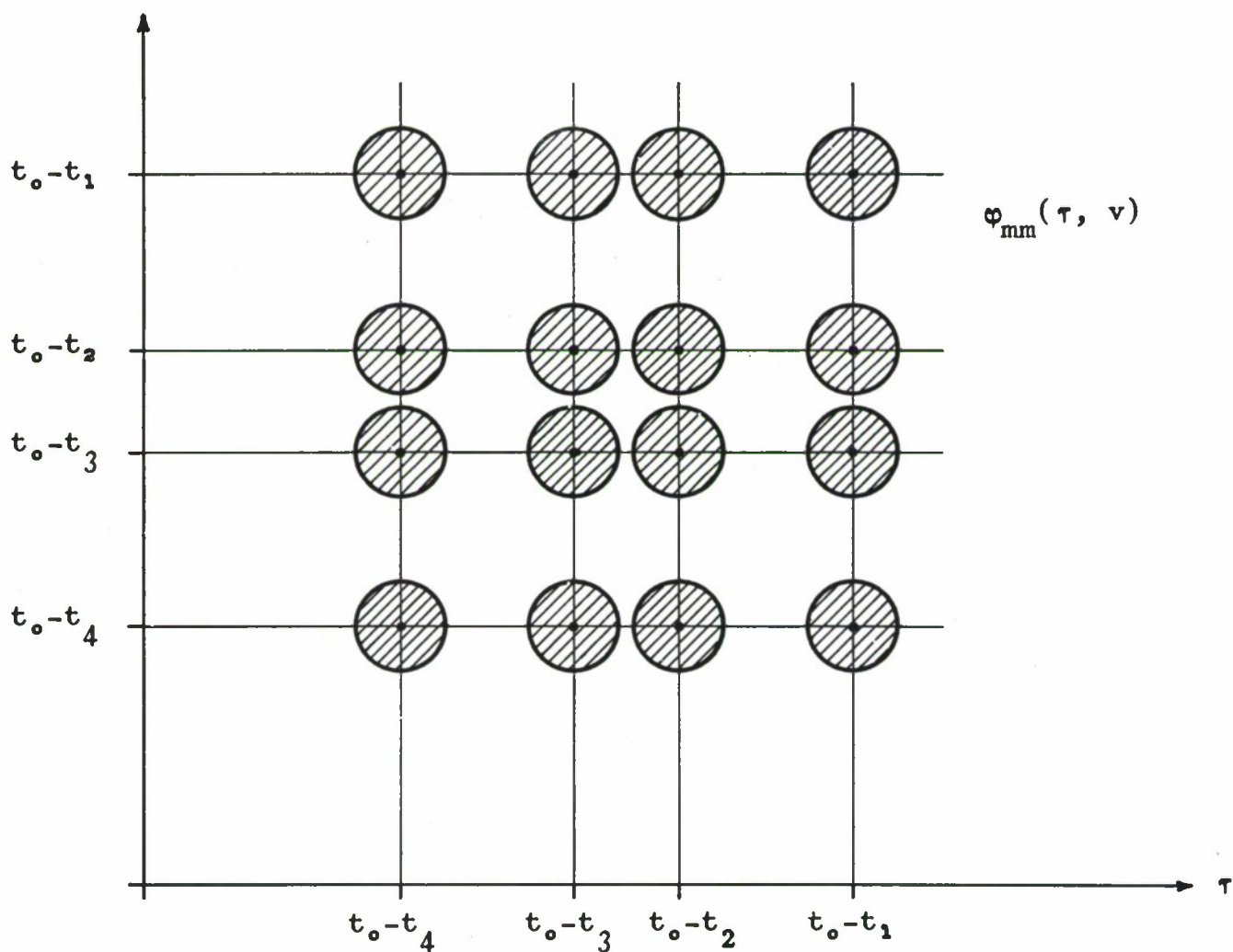

FIG. 4b   THE CORRELATION FUNCTION $\varphi_{mm}(\tau, v)$
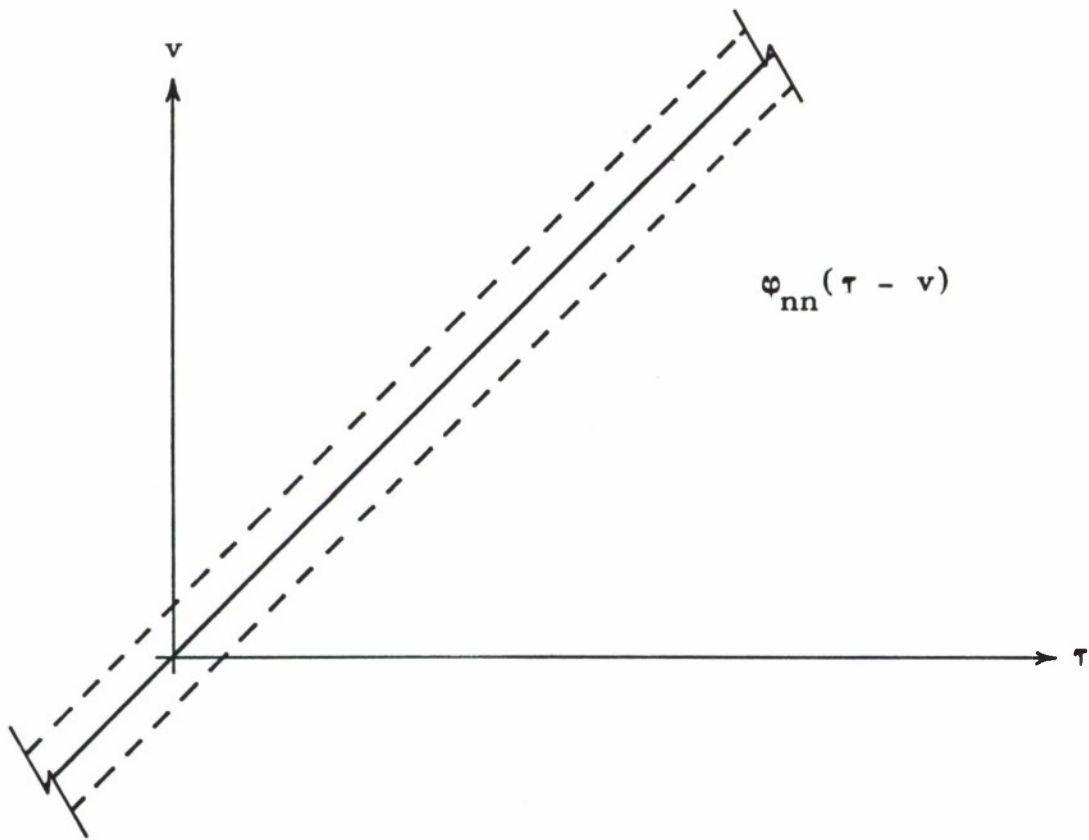
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FIG. 4c   THE CORRELATION FUNCTION $\varphi_{nn}(\tau - v)$
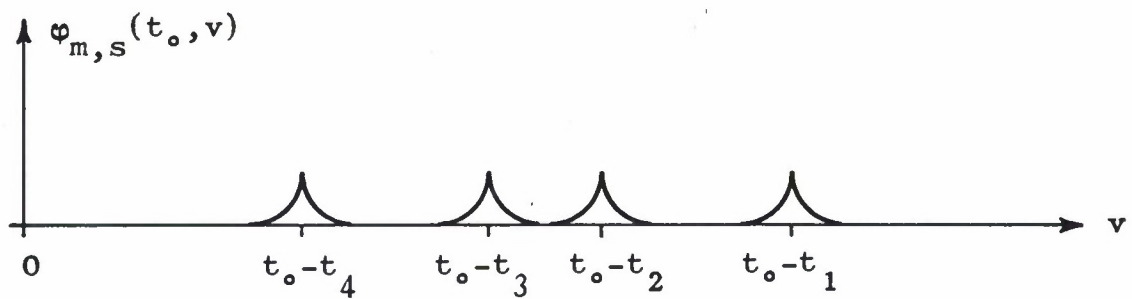
FIG. 4d   THE CORRELATION FUNCTION $\varphi_{m,s}(t_o,v)$

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Eq. 7 may be transformed to yield

$$\int_{-\infty}^{+\infty} G(f) \cdot \Phi_{m.m}(f,\xi) \, df + T \cdot G(\xi) \cdot N(\xi) = \Phi_{m.s}(t_o,\xi) \, .$$

Introducing finally

$$F_{m\,m}(f,\xi) = \Phi_{m\,m}(f,\xi)/T \cdot N(\xi)$$

and

$$F_{m.s}(t_o,\xi) = \Phi_{m.s}(t_o,\xi)/T \cdot N(\xi) \, ,$$

one obtains

$$\int_{-\infty}^{+\infty} G(f) \cdot F_{m.m}(f,\xi) \, df + G(\xi) = F_{m.s}(t_o,\xi) \, . \qquad \text{(Eq. 8)}$$

This will be recognized as an inhomogeneous integral equation of the second kind.

3.   A PRACTICAL EXTRACTION TECHNIQUE

The theoretical treatment of the extraction problem presented
in Ch. 2 can hardly be considered very practical for echo-ranging
situations, because the signal and path information required for
the functions $F_{m.m}(f, \xi)$ and $F_{m.s}(t_o, \xi)$ of Eq. 8 will rarely be
available. In this chapter we shall attempt to solve the extraction
problem for an echo-ranging situation involving time-invariant
transmission paths by means of a more practical intuitive approach.

The situation to be considered is similar to the one depicted in Fig.
2c. It is assumed that the paths are time-invariant, and that,
except for the path delays which may be different, they have
identical and ideal transmission characteristics. Such conditions
may be encountered when the ocean surface is very calm. Furthermore,
it is assumed that the points of transmission and reception are
the same. Paths D-S and S-D will thus coincide and the impulse
response of the path combination between points a and b in Fig. 2c
is as shown in Fig. 5a. The absolute value of the Fourier
transform H(f) of this impulse response is as shown in Fig. 5b.
It is easily shown that

$$H(f) = -2(1 - \cos 2\pi \cdot f \cdot \Delta t) \cdot e^{-j2\pi ft_2} , \qquad (Eq. 9a)$$

where Δt is the separation between successive paths and t_2 is
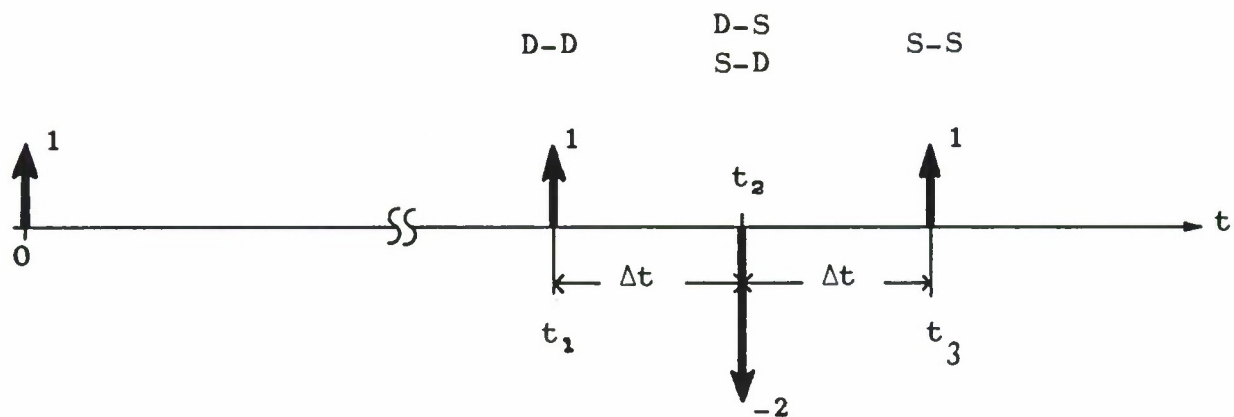the delay associated with the D-S/S-D paths. Note that the

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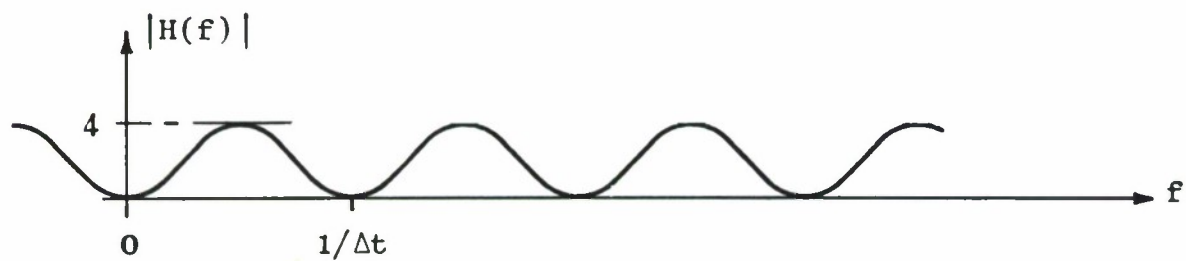FIG. 5a   IMPULSE RESPONSE OF MULTIPATH CHANNEL

FIG. 5b   FREQUENCY TRANSFER FUNCTION OF MULTIPATH CHANNEL

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pattern of Fig. 5a is symmetric about the D-S/S-D paths and that the responses due to these are inverted due to one surface reflection. The time delay expressed by the term $-e^{-j2\pi ft_2}$ in $H(f)$ will be ignored, since this is of no importance for what follows. Thus we shall write

$$H(f) = 2(1 - \cos 2\pi f \cdot \Delta t) \qquad \qquad \text{(Eq. 9b)}$$

The idea behind the extraction technique will be described by reference to Fig. 6a. A signal $s(t)$ equal to the impulse response of the target is the input to a linear filter with impulse response equal to the multipath structure shown in Fig. 5a. Gaussian noise $n(t)$ is added to the multipath signal and the mixture fed to the inverse filter, the structure of which is to be determined. The output of this filter can be considered to be the sum of three components. The first is a desired signal $k \cdot s(t)$. The second is undesired signal components $s_m(t)$. The third is the output $n_o(t)$ due to the noise $n(t)$. The three components are indicated in Fig. 6b. The design criterion for the inverse filter is that it should maximize the ratio between the energy of the desired signal and the energy, in the same interval, of the undesired signal components and noise. Thus one wants to maximize the ratio

$$r = \frac{k^2 \displaystyle\int_{t_o}^{t_o+T} s^2(t)\, dt}{E\left(\displaystyle\int_{t_o}^{t_o+T} \left[s_m(t) + n_o(t) \right]^2 dt \right)} \quad ,$$
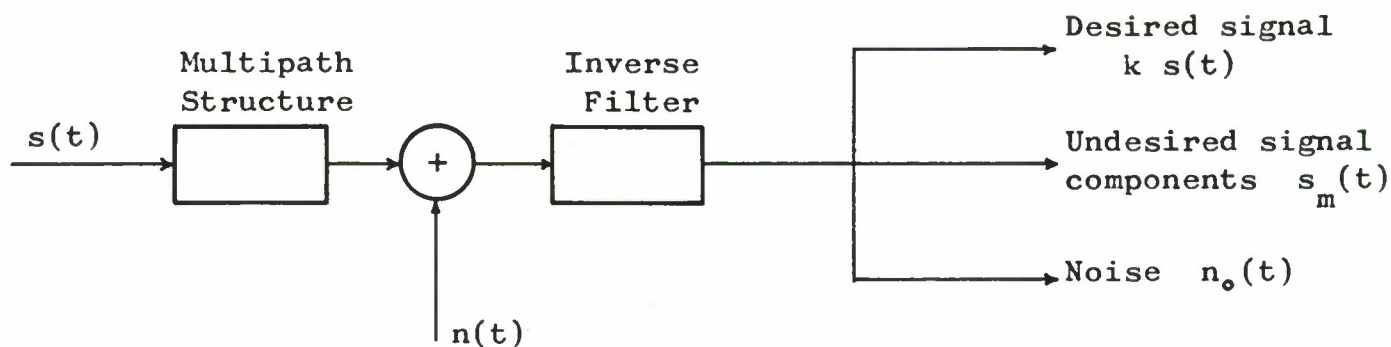
FIG. 6a EQUIVALENT DIAGRAM OF MULTIPATH/INVERSE-FILTER COMBINATION
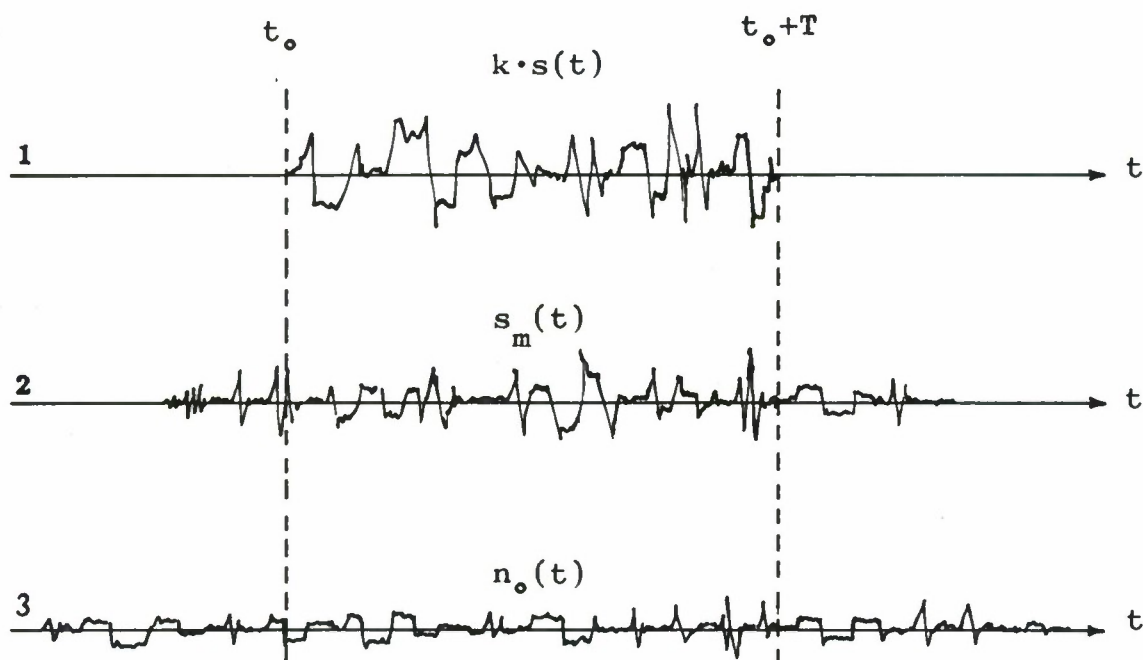
FIG. 6b THE COMPONENTS AT THE OUTPUT OF THE INVERSE FILTER

where the statistical averaging indicated by $E(\quad)$ applies to the noise $n_o(t)$. Since the noise will be statistically independent of $s_m(t)$ and will be assumed to have zero mean value, one can write

$$r = \frac{k^2 \cdot P_s}{\frac{1}{T} \int_{t_o}^{t_o+T} s^2_m(t) \, dt + E\left(n_o^2(t)\right)} \quad , \qquad \text{(Eq. 10)}$$

where

$$P_s = \frac{1}{T} \int_{t_o}^{t_o+T} s^2(t) \, dt \ .$$

We shall first consider the frequency function $H_i(t)$ of the ideal inverse filter when $n(t) = 0$, that is for the noiseless situation. The form of this filter is

$$H_i(f) = 1/H(t) \ ,$$

with $H(f)$ given by Eq. 9 as

$$H(f) = 2(1 - \cos 2\pi f \cdot \Delta t) \ .$$

Thus

$$H_i(f) = 1/2 \ (1 - \cos 2\pi f \cdot \Delta t) \ .$$

The functions H(f) and H$_i$(f) are shown in Fig. 7. The product
of the two is of course equal to unity for all f and is the desired
total response yielding an output equal to s(t).
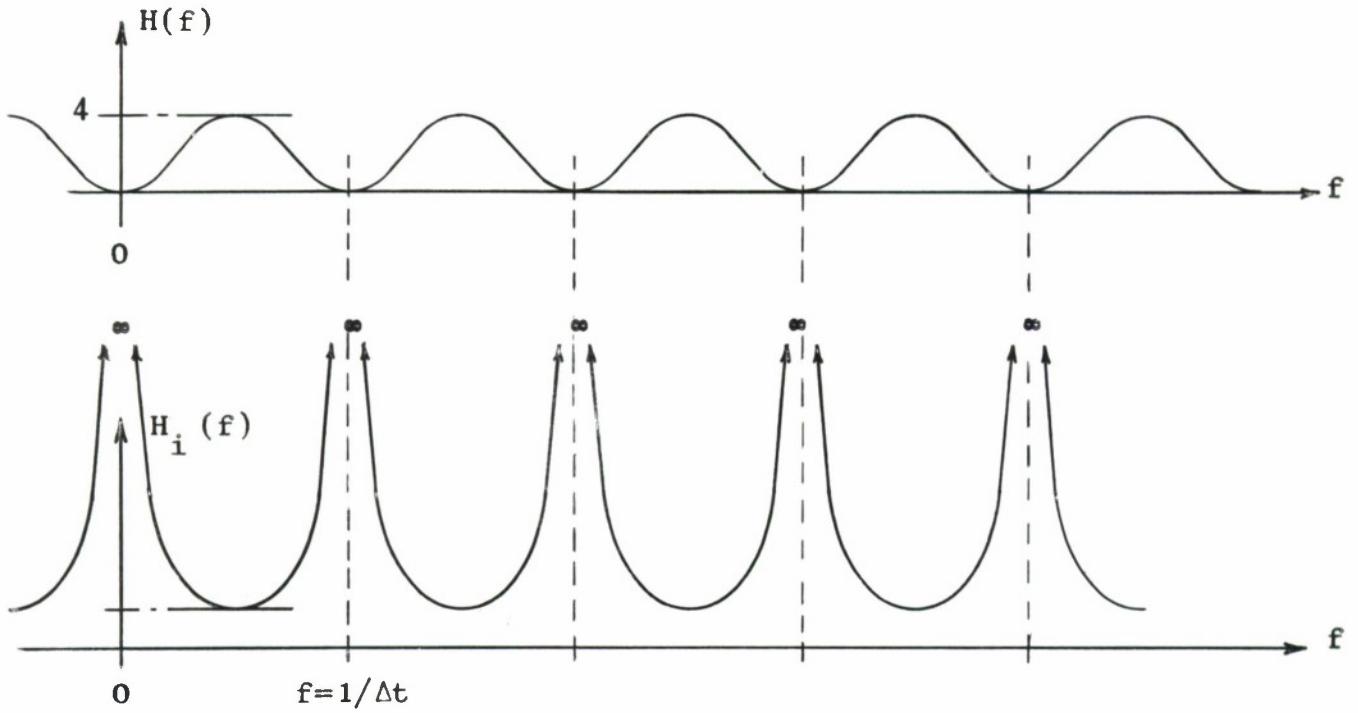
FIG. 7 THE IDEAL INVERSE FILTER n(t) = 0

Consider next the more practical situation with n(t) ≠ 0. We
shall assume that n(t) has uniform power spectral density.
Referring to Fig. 7 it will be apparent that the inverse filter,
with its large amplitudes at frequencies that are multiples of
1/Δt, will introduce noise of very high intensity at the output.
It is therefore necessary to modify the inverse filter in a
direction that will tend to maximize the ratio r of Eq. 10.

Figure 8 illustrates one possible modification. It is seen that $H_i(f)$ is periodic with period $1/\Delta t$.
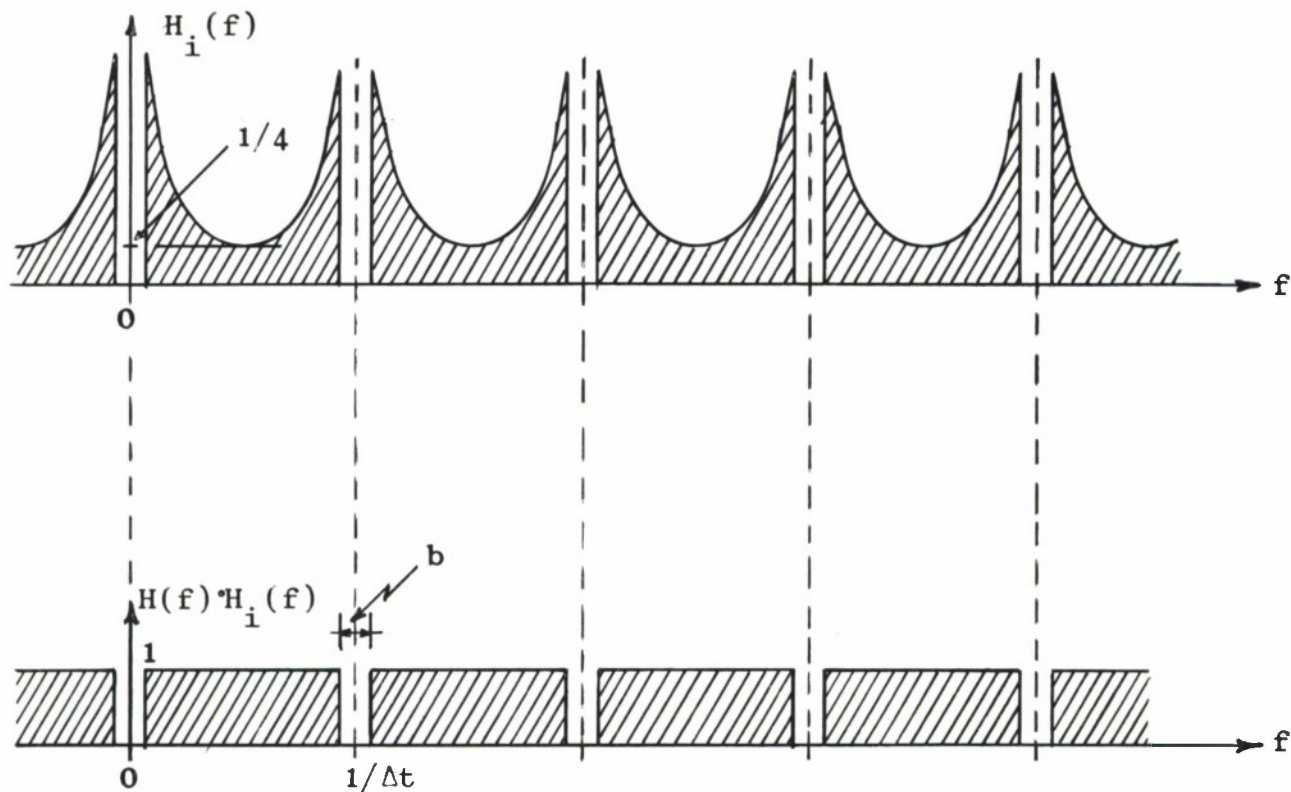
FIG. 8 MODIFIED INVERSE FILTER FUNCTION

For $f < 1/2\Delta t$ we have

$$H_i(f) = \begin{cases} 1/2 \ (1 - \cos 2\pi f.\Delta t) & \text{for} \quad \dfrac{b}{2} < |f| < 1/2 \ \Delta t \\[2ex] 0 \quad \text{for} \quad |f| < \dfrac{b}{2} \end{cases}$$

(Eq. 11)

The problem is now to determine b such that the ratio r of
Eq. 10 is maximum. It should be noted that this is only one
possible form of the inverse filter. Other modifications involving
functions with less abrupt changes may be more useful.

Figure 8 also shows the total frequency response $H(f) \cdot H_i(f)$. This
may be decomposed into the two components shown in Fig. 9.

FIG. 9 DECOMPOSITION OF $H(f) \cdot H_i(f)$

Thus

$$H(f) \cdot H_i(f) \quad = \quad 1 - H_m(f) \; . \tag{Eq. 12}$$

For practical purposes it is necessary to impose bandwidth
limitations on the inverse filter or the receiving circuits
preceding it. We shall assume that the signal s(t) contains no

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frequency components at frequencies greater than $f = B$, where

$B \gg 1/\Delta t$,

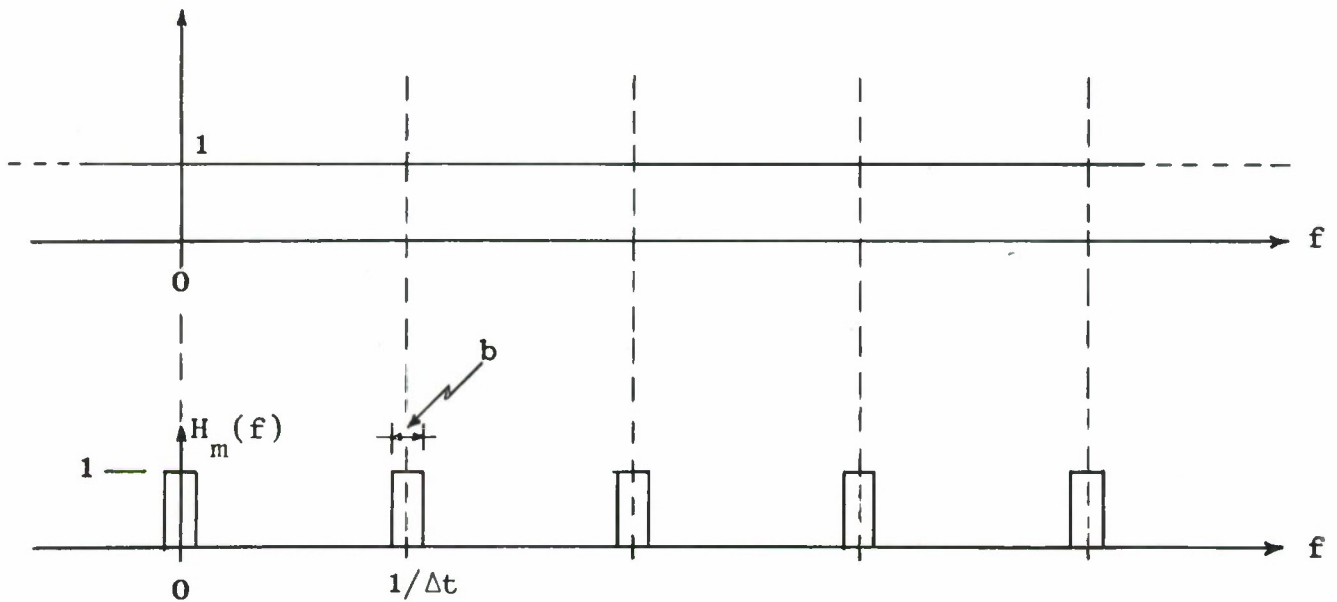and that the receiving circuit preceding the inverse filter has the characteristics of an ideal lowpass filter of bandwidth B. If the input noise power in this frequency band is P_N it is easily shown with the aid of Appendix B that the signal-to-noise ratio r of Eq. 10 becomes

$$r = \frac{P_S \cdot (1 - a_o)^2}{\sum\limits_{i=1}^{\infty} \sum\limits_{j=1}^{\infty} C_{ij} \cdot a_i \cdot a_j + \frac{P_N}{8\pi} \left[\tan \frac{\pi}{2} (1 - b \cdot \Delta t) + \frac{1}{3} \tan^3 \frac{\pi}{2} (1 - b \cdot \Delta t) \right]} \qquad (\text{Eq. } 13)$$

where

$$P_S = \frac{1}{T} \int_{t_o}^{t_o + T} s^2(t) \, dt$$

and

$$C_{ij} = \frac{1}{T} \int_{t_o}^{t_o + T} \left[s(t - t_o - i \cdot \Delta t) + s(t - t_o + i \cdot \Delta t) \right] \left[s(t - t_o - j \cdot \Delta t) + s(t - t_o + j \cdot \Delta t) \right]$$

and

a_o, a_i are the zero'th and i'th coefficient respectively in the
Fourier Series expansion of $H_m(f)$. (See Appendix B.)
They depend on b.

The signal-to-noise ratio as given by Eq. 13 must be maximized with respect to b. This process can generally be greatly simplified when one further assumption can be made. Thus if

$$C_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ \\ 0 & \text{for} \quad i \neq j \end{cases} \qquad \text{(Eq. 14)}$$

we can write

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} C_{ij} \cdot a_i \cdot a_j = P_s \sum_{i=1}^{\infty} a_i^2 \quad .$$

According to Appendix B

$$a_o = b \cdot \Delta t$$

and

$$\sum_{i=1}^{\infty} a_i^2 = b \cdot \Delta t (1 - b \cdot \Delta t).$$

Therefore

$$r = \frac{\dfrac{P_s}{P_N} (1 - b \cdot \Delta t)^2}{\dfrac{P_s}{P_N} \cdot b \cdot \Delta t (1 - b \cdot \Delta t) + \dfrac{1}{8 \cdot \pi} \left[\tan \dfrac{\pi}{2} (1 - b \cdot \Delta t) + \dfrac{1}{3} \tan^3 \dfrac{\pi}{2} (1 - b \cdot \Delta t) \right]} \qquad \text{(Eq. 15)}$$

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Equation 15 can be differentiated with respect to b in order to determine the value $b = b_o$ yielding the maximum value of r. The value of b_o will clearly depend on the ratio P_S/P_N, as will also r_{max}.

In order that the assumption contained in Eq. 14 shall be valid it is necessary for the signal bandwidth B_s to be much greater than $1/\Delta t$ and for the signal length T to be much greater than $1/b_o$. The condition $T \gg 1/b_o$ ensures that the major part of the components of $s_m(t)$ of Figs. 5 and 6 will fall within the time-interval $\left[t_o, (t_o + T) \right]$. C_{ij} will then involve auto-correlation functions of $s(t)$ with arguments $(i \pm j) \Delta t$. Since the minimum value of $(i \pm j) \cdot \Delta t$, $i \neq j$, is Δt, the condition $B_s \gg 1/\Delta t$ implies that these auto-correlation functions are zero except when $i = j$.

4. COMMENTS

An extraction technique following the very general theoretical prescriptions of Ch. 2 is not considered very practical due to the amount of a priori knowledge required about channel and signal (target) statistics and also due to difficulties in solving the integral equation involved.

The less general technique outlined in Ch. 3 appears to be of greater practical value. However, further studies are required before the true value of this approach can be properly assessed. These studies should relate to the accuracies with which the quantities P_S/P_N and Δt of Eqs. 13 or 15 can be estimated and to the influence of estimation errors on the mean square error of the extracted signal.

APPENDIX A

THE MINIMIZATION PROCESS

It is desired to find the physically-realizable filter impulse
response $g(\tau)$ that minimizes the expression

$$\mu_g = E\left(\int_{t_o}^{t_o+T} \left[\int_0^\infty g(\tau) \cdot y(t-\tau)\ d\tau - s(t-t_o) \right]^2\ dt \right) \qquad \text{Eq. A.1}$$

Inserting

$$k(\tau) = g(\tau) + \varepsilon \cdot \eta(\tau) \qquad \text{Eq. A.2}$$

for $g(\tau)$ one obtains

$$\mu_k = E\Bigg(\int_{t_o}^{t_o+T} \int_0^\infty \int_0^\infty k(\tau) \cdot k(v) \cdot y(t-\tau) \cdot y(t-v)\ dv \cdot d\tau \cdot dt$$

$$-2 \int_{t_o}^{t_o+T} s(t-t_o) \cdot \int_0^\infty k(\tau) \cdot y(t-\tau)\ d\tau \cdot dt + \int_{t_o}^{t_o+T} s^2(t-t_o)\ dt \Bigg)$$

Eq. A.3

After substitution for $h(\tau)$ from Eq.(A.2) it is desired to evaluate the differential $\partial \mu_k/\partial \epsilon$ at $\epsilon = 0$.

$$\left(\frac{\partial}{\partial} \frac{\mu k}{\epsilon}\right)_{\epsilon=0} = 2 \int_0^\infty \eta(v) \left[\int_0^\infty g(\tau) \cdot \int_{t_o}^{t_o+T} \cdot E\Big(y(t-\tau) \cdot y(t-v)\Big) \, dt \cdot d\tau \right.$$

$$\left. - \int_{t_o}^{t_o+T} s(t-t_o) \cdot E\Big(y(t-v)\Big) \, dt \right] dv$$

It is required that this be zero for all $\eta(v)$. This is obtained when the quantity inside the square parenthesis is zero. Thus

$$\int_0^\infty g(\tau) \cdot \int_{t_o}^{t_o+T} E\Big(\cdot y(t-\tau) \cdot y(t-v)\Big) \, dt \quad d\tau$$

$$- \int_{t_o}^{t_o+T} s(t-t_o) \cdot E\Big(y(t-v)\Big) \, dt \quad = 0 \qquad\qquad \text{Eq. A.4}$$

$$\text{for} \quad v > 0$$

Since $y(t)$ is the sum of a multipath signal $m(t)$ and stationary noise $n(t)$, we can introduce the functions

$$\varphi_{m.m}(\tau,v) = \int_{t_o}^{t_o+T} E\Big(m(t-\tau) \cdot m(t-v)\Big) \, dt$$

$$\varphi_{n.n}(\tau - v) = E\left(n(t-\tau) \cdot n(t-v)\right)$$

$$\varphi_{m.s}(t_o, v) = \int_{t_o}^{t_o + T} s(t-t_o) \cdot E\left(m(t-v)\right) dt$$

into Eq.(A.4). The result is

$$\int_o^\infty g(\tau) \left[\varphi_{m.m}(\tau, v) + T \cdot \varphi_{n.n}(\tau - v)\right] d\tau - \varphi_{m.s}(t_o, v) = 0$$

$$\text{for} \quad v > 0$$

APPENDIX B

THE OUTPUT SIGNAL-TO-NOISE RATIO

The total response of the channel-filter combination was given by Eq. 12 of Ch. 3 as

$$H(f) \cdot H_i(f) \quad = \quad 1 - H_m(f) \qquad\qquad \text{Eq. B.1}$$

We shall study this response function in the time domain and separate it into two components, one giving rise to the desired signal and the other to undesired signal components $s_m(t)$.

Consider first the function $H_m(f)$ depicted in Fig. B.1. It is periodic, with period $1/\Delta t$, and can be expressed by the Fourier series

$$H_m(f) = a_0 + 2 \sum_{i=1}^{\infty} a_i \cdot \cos 2\pi \cdot i \cdot \Delta t \ f \qquad\qquad \text{Eq. B.2}$$

It can be readily shown that

$$a_0 = \Delta t \int_{-1/2 \cdot \Delta t}^{1/2\Delta t} H_m(f) \ df = b \cdot \Delta t \qquad\qquad \text{Eq. B.3}$$

and that

$$\Delta t \int_{-1/2 \cdot \Delta t}^{1/2\Delta t} \left| H_m(f) \right|^2 df = a_0^2 + 2 \sum_{i=1}^{\infty} a_i^2 \ .$$

The latter yields

$$b \cdot \Delta t = a_o^2 + 2 \sum_{i=1}^{\infty} a_i^2$$

or

$$2 \sum_{i=1}^{\infty} a_i^2 = b \cdot \Delta t \cdot (1 - b \cdot \Delta t) .$$

Eq. B.4

We return now to Eq. B.1. Inserting for $H_m(f)$ from Eq. B.2 gives

$$H(f) \cdot H_i(f) = (1 - a_o) - 2 \sum_{i=1}^{\infty} a_i \cos 2\pi \cdot i \cdot \Delta t \cdot f .$$

Performing a Fourier transform, we obtain the impulse response $h_{\Sigma}(t)$ of the channel-filter combination. This function is illustrated in Fig. B.1 and is

$$h_{\Sigma}(t) = (1 - a_o) \delta(t) - \sum_{i=1}^{\infty} a_i \left[\delta(t - i \cdot \Delta t) + \delta(t + i \cdot \Delta t) \right]$$

Eq. B.5

Here the term $(1 - a_o) \delta(t)$ carries the desired signal component. The remaining terms carry undesired signals $s_m(t)$.
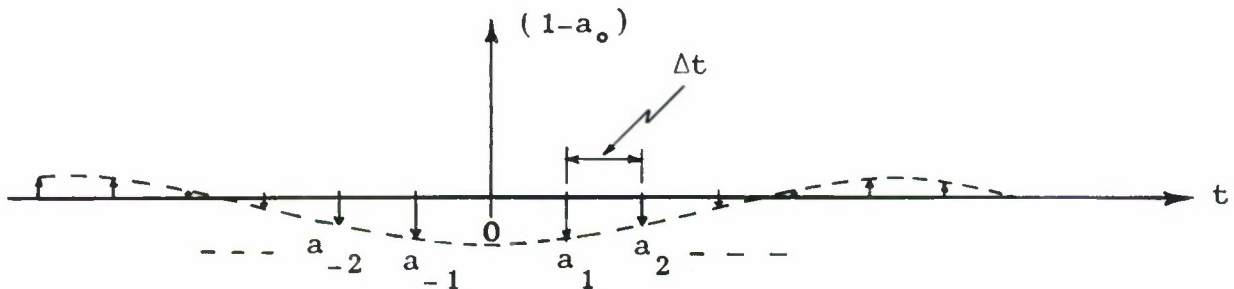

FIG. B.1 THE IMPULSE RESPONSE $h_{\Sigma}(t)$

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In the derivation of Eq. B.5 , time-delays in the channel and filter response functions have purposely been omitted on the grounds that they do not affect the magnitudes of the quantities considered in Eq. 10. A real situation involves a channel delay, which in Eq. 9a is denoted by t_2, and a filter delay t_f which is necessary if the inverse filter is to be physically realizable. The delay t_f must be approximately half the extension of the Fourier transform of $H_i(f)$ in the time domain. Writing

$$t_o = t_2 + t_f ,$$

Eq. B.5 should be written

$$h_\Sigma(t) = (1-a_o)\,\delta(t-t_o) - \sum_{i=1}^{\infty} a_i \left[\delta(t-t_o-i\cdot\Delta t) + \delta(t-t_o+i\cdot\Delta t) \right] \qquad \text{Eq. B.6}$$

If we apply a signal $s(t)$ to the channel input at time $t = 0$, the filter output signal becomes

$$u(t) = (1-a_o)\ s(t-t_o) + s_m(t) ,$$

where

$$s_m(t) = -\sum_{i=1}^{\infty} a_i \left[s(t-t_o-i\cdot\Delta t) + s(t-t_o+i\cdot\Delta t) \right] .$$

We are now able to extract the quantities required for Eq. 10. Thus

$$k^2 = (1 - a_o)^2 \qquad\qquad\qquad \text{Eq. B.7}$$

$$\frac{1}{T} \int_{t_o}^{t_o+T} s_m^2(t)\, dt = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} \cdot a_i \cdot a_j \ ,$$

where

$$C_{ij} = \frac{1}{T} \int_{t_o}^{t_o+T} \Big[s(t-t_o-i\cdot\Delta t) + s(t-t_o+i\cdot\Delta t) \Big]\Big[s(t-t_o-j\cdot\Delta t) + s(t-t_o+j.\Delta t) \Big] \ dt$$

$$\text{(B.8)}$$

The final information needed for Eq. 10 is the evaluation of the noise term $E\left(n_o^2(t)\right)$. We can write this as

$$E\left(n_o^2(t)\right) = \frac{P_N}{B} \int_0^B |H_i(f)|^2 \ df \ .$$

Since $H_i(f)$ is periodic, with period $1/\Delta t$ and $B \gg 1/\Delta t$, we can write

$$E\left(n_o^2(t)\right) = 2 \cdot \frac{P_N}{B} \cdot B \cdot \Delta t \int_0^{1/2\cdot\Delta t} |H_i(f)|^2 \ df \ .$$

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Inserting $H_i(f) = 1/2 (1 - \cos 2\pi \cdot f \cdot \Delta t)$

and performing the integration, yields

$$E\left(n_o^2(t)\right) = \frac{P_N}{8 \cdot \pi} \left[\tan \frac{\pi}{2}(1 - b \cdot \Delta t) + \frac{1}{3} \tan^3 \frac{\pi}{2}(1 - b \cdot \Delta t) \right] . \qquad \text{Eq. B.9}$$

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